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SIMPLIFICATIONS UNDER THE KIRCHHOFF HYPOTHESIS OF TABER'S NONLINEAR THEORY FOR THE AXISYMMETRIC BENDING AND TORSION OF ELASTIC SHELLS OF REVOLUTION

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Abstract—We show that, under the Kirchhoff hypothesis, Taber's recent theory for the simultaneous axisymmetric bending and torsion of shells of revolution undergoing large strains can be simplified considerably. In general, his 33 equations can be reduced to four first-order ordinary differential equations and two algebraic equations for six unknowns. For small strains, the equations can be reduced further to two coupled nonlinear equations for the meridional angle of rotation and a stress function, as in Reissner's theory of torsionless, axisymmetric deformation.

1. INTRODUCTION

Taber (1988) has developed a theory of circumferentially complete, rubber-like (isotropic, nonlinearly elastic) shells of revolution under static surface and end loads that are independent of the polar angle. Introducing one-dimensional extensional, bending, and transverse shearing strains, Taber derived a set of 33 field equations for 33 unknowns. In the present paper, we show that under the reduced Kirchhoff hypothesis—which assumes that the two-dimensional transverse shear strain–twist vector (Libai and Simmonds, 1983, p. 314) vanishes—the field equations can be reduced to a coupled system of four first-order ordinary differential equations plus two algebraic equations for six unknowns. (For simplicity, we omit surface loads.) Three of the unknowns are the same as Taber's, namely $\bar{\alpha}$, the angle a tangent to a meridian of the deformed reference surface makes with the horizontal; κ, a meridional bending strain; and $\chi = \psi'$, the derivative of the polar twist with respect to undeformed meridional arc length. [Reissner (1968) earlier introduced the two angles $\bar{\alpha}$ and ψ (calling them Φ and Θ) in a study of the helical *inextensional* bending and torsion of incomplete shells of revolution.] The other three unknowns in our formulation are a meridional and a hoop stretch, Λ_s and Λ_θ, and a stress function, F. The latter is a standard unknown in the theory of axishells—shells of revolution undergoing *torsionless* axisymmetric deformation (Reissner, 1950; Libai and Simmonds, 1988).

If the strains are small enough to justify the introduction of a quadratic strain-energy density, we may give the field equations a particularly simple and symmetric form by first expressing the meridional and hoop stretches in terms of F and a second stress function, G; and second, by dropping terms in the resulting equations similar to those we ignore implicitly when we adopt a quadratic strain-energy density. These arguments are like those used by Simmonds and Libai (1987) and Libai and Simmonds (1988) to simplify Reissner's axishell equations.

If linearized, the field equations for $\bar{\alpha}$ and F uncouple from those for χ and G and the well-known static-geometric duality of linear theory manifests itself in the pairings

$$\beta \leftrightarrow F, \quad -\tfrac{1}{2}r\chi\sin\alpha \leftrightarrow G, \tag{1}$$

where

$$\beta \equiv \bar{\alpha} - \alpha \tag{2}$$

is the angle of rotation of a deformed meridian.

A detailed development of *linear* theories for elastically isotropic or anisotropic shells of revolution suffering bending and torsion is given by Reissner and Wan (1969, 1971).

2. KINEMATICS OF DEFORMATION

Let

$$\mathcal{S}: \mathbf{y} = r(s)\mathbf{e}_r(\theta) + z(s)\mathbf{e}_z, \quad s_1 \leqslant s \leqslant s_2, \quad 0 \leqslant \theta \leqslant 2\pi \tag{3}$$

denote the parametric representation of the reference surface of revolution. Here, s is arc length along a meridian and (r, θ, z) are circular cylindrical coordinates with associated orthonormal base vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$, related in the usual way to a fixed right-handed Cartesian reference frame $Oxyz$ with associated orthonormal base vectors $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. In particular,

$$\mathbf{e}_r = \mathbf{e}_x \cos\theta + \mathbf{e}_y \sin\theta, \quad \mathbf{e}_\theta = -\mathbf{e}_x \sin\theta + \mathbf{e}_y \cos\theta. \tag{4}$$

We assume that the loading and material properties of the shell are such that the deformed reference surface† has the representation

$$\bar{\mathcal{S}}: \bar{\mathbf{y}} = \bar{r}(s)\bar{\mathbf{e}}_r(s, \theta) + \bar{z}(s)\mathbf{e}_z. \tag{5}$$

Here (and for future reference),

$$\bar{\mathbf{e}}_r \equiv \mathbf{e}_r(\theta + \psi(s)), \quad \bar{\mathbf{e}}_\theta \equiv \mathbf{e}_\theta(\theta + \psi(s)), \tag{6a, b}$$

where ψ is one of our basic unknowns.

Associated with the parametric representation (3) of the reference surface of revolution are the standard covariant surface base vectors

$$\mathbf{y}_{,s} \equiv \mathbf{t}(s, \theta) = r'(s)\mathbf{e}_r(\theta) + z'(s)\mathbf{e}_z, \quad \mathbf{y}_{,\theta} = r(s)\mathbf{e}_\theta(\theta), \tag{7}$$

and the surface normal

$$\mathbf{n} = \mathbf{t} \times \mathbf{e}_\theta, \tag{8}$$

where a subscript preceded by a comma denotes partial differentiation with respect to that subscript. Because s is arc length along a meridian, we may set

$$\cos\alpha \equiv r'(s), \quad \sin\alpha \equiv z'(s). \tag{9}$$

Thus, $\mathbf{t}$ and $\mathbf{n}$ are unit vectors.

Let

$$\begin{aligned} \hat{\mathbf{t}} &\equiv \frac{\bar{r}'(s)\bar{\mathbf{e}}_r(s, \theta)}{\sqrt{\bar{r}'^2(s) + \bar{z}'^2(s)}} + \frac{\bar{z}'(s)\mathbf{e}_z}{\sqrt{\bar{r}'^2(s) + \bar{z}'^2(s)}} \\ &\equiv \cos\bar{\alpha}(s)\bar{\mathbf{e}}_r(s, \theta) + \sin\bar{\alpha}(s)\mathbf{e}_z \end{aligned} \tag{10}$$

denote a unit tangent to a meridian of $\bar{\mathcal{S}}$. (In general, $\hat{\mathbf{t}}$ is *not* the deformed image of $\mathbf{t}$.) Then the covariant base vectors and unit normal of the deformed reference surface, $\bar{\mathcal{S}}$, are, by (5) and (6),

† Here, we follow Simmonds (1979) and Libai and Simmonds (1983) and take the position of the reference surface of a shell, $\mathcal{S}$, and its deformed image, $\bar{\mathcal{S}}$, to be density weighted averages of the initial and final three-dimensional positions of the shell. Thus, $\mathcal{S}$ and $\bar{\mathcal{S}}$ need not comprise the same particles.

$$\bar{\mathbf{y}}_{,s} = \bar{r}'(s)\bar{\mathbf{e}}_r(s,\theta) + \bar{r}(s)\psi'(s)\bar{\mathbf{e}}_\theta(s,\theta) + \bar{z}'(s)\mathbf{e}_z$$
$$\equiv \Lambda_s(s)\hat{\mathbf{t}}(s,\theta) + \Gamma(s)\bar{\mathbf{e}}_\theta(s,\theta) \tag{11}$$

$$\bar{\mathbf{y}}_{,\theta} = \bar{r}(s)\bar{\mathbf{e}}_\theta(s,\theta) \equiv r(s)\Lambda_\theta(s)\bar{\mathbf{e}}_\theta(s,\theta) \tag{12}$$

$$\bar{\mathbf{n}} = \hat{\mathbf{t}}(s,\theta)\times\bar{\mathbf{e}}_\theta(s,\theta) = -\sin\bar{\alpha}(s)\bar{\mathbf{e}}_r(s,\theta) + \cos\bar{\alpha}(s)\mathbf{e}_z. \tag{13}$$

Here, Λ_s and Λ_θ are meridional and hoop stretches and Γ is an in-surface shear strain.

Bending strains will be introduced presently as a natural consequence of the Principle of Virtual Work.

3. STRETCH AND STRAIN COMPATIBILITY

Assuming sufficient smoothness, we must have $\bar{\mathbf{y}}_{,s\theta} = \bar{\mathbf{y}}_{,\theta s}$ or, by (6) and (10)–(12),

$$(r\Lambda_\theta)' = \Lambda_s \cos\bar{\alpha} \tag{14}^*$$

$$\Gamma = r\Lambda_\theta\psi'. \tag{15}$$

(Following the suggestion of a referee, we have indicated each member of our final set of field equations by an asterisk.)

4. FORCE AND MOMENT EQUILIBRIUM

By specializing the three-dimensional integral equations of force and moment equilibrium to a shell-like volume, one may obtain exact two-dimensional integral equations over a reference surface (Simmonds, 1979; Libai and Simmonds, 1983). If the stress resultant and couple tensors in these integral equations are sufficiently smooth, as we shall assume, then the divergence theorem may be applied to obtain differential equations of equilibrium. For the simultaneous axisymmetric bending and torsion of a shell of revolution, free of surface loads, these equations take the form

$$(r\mathbf{N}_s)_{,s} + \mathbf{N}_{\theta,\theta} = \mathbf{0} \tag{16}$$

$$(r\mathbf{M}_s)_{,s} + \mathbf{M}_{\theta,\theta} + r\bar{\mathbf{y}}_{,s}\times\mathbf{N}_s + \bar{\mathbf{y}}_{,\theta}\times\mathbf{N}_\theta = \mathbf{0}. \tag{17}$$

Here, $r\mathbf{N}_s\,d\theta$ and $\mathbf{N}_\theta\,ds$ are, respectively, the forces acting across the deformed images of the differential coordinate elements, $r\,d\theta$ and ds; $r\mathbf{M}_s\,d\theta$ and $\mathbf{M}_\theta\,ds$ are analogous couples.

We now take the dot product of (16) with $\delta\bar{\mathbf{y}}$, the variation of the deformed position in (5), and the dot product of (17) with $\delta\boldsymbol{\omega}$ a yet-to-be-defined unknown. Adding the resulting equations and integrating by parts over $\mathscr{S}$ to remove partial derivatives on the stress resultants and couples, we arrive at the identity

$$EVW \equiv IVW, \tag{18}$$

where

$$EVW = 2\pi[r(\mathbf{N}_s\cdot\delta\bar{\mathbf{y}} + \mathbf{M}_s\cdot\partial\boldsymbol{\omega})]_{S_1}^{S_2} \tag{19}$$

is the *external virtual work* and

$$IVW = 2\pi \int_{S_1}^{S_2} [r\mathbf{N}_s \cdot (\delta \bar{\mathbf{y}}_{,s} - \delta\omega \times \bar{\mathbf{y}}_{,s}) + \mathbf{N}_\theta \cdot (\delta \bar{\mathbf{y}}_{,\theta} - \delta\omega \times \bar{\mathbf{y}}_{,\theta}) + r\mathbf{M}_s \cdot \delta\omega_{,s} + \mathbf{M}_\theta \cdot \delta\omega_{,\theta}]\, ds \tag{20}$$

is the *internal virtual work*.

Now by (6) and (10)–(13),

$$\delta \bar{\mathbf{y}}_{,s} = \hat{\mathbf{t}}\delta\Lambda_s + \bar{\mathbf{e}}_\theta \delta\Gamma + (\delta\omega + \delta\gamma) \times \bar{\mathbf{y}}_{,s}, \quad \delta \bar{\mathbf{y}}_{,\theta} = r\bar{\mathbf{e}}_\theta \delta\Lambda_\theta + (\delta\omega + \delta\gamma) \times \bar{\mathbf{y}}_{,\theta}, \tag{21}$$

where

$$\delta\gamma \equiv \mathbf{e}_z \delta\psi(s) - \bar{\mathbf{e}}_\theta(s, \theta)\delta\bar{\alpha}(s) - \delta\omega \tag{22}$$

is a virtual transverse shear strain–twist vector (Libai and Simmonds, 1983, p. 314). The *reduced Kirchhoff Hypothesis* is that the strain-energy density of the shell does not depend on γ. (The adjective "reduced" is used to distinguish this *two-dimensional* hypothesis from the classical Kirchhoff Hypothesis in which the *three-dimensional* transverse shearing and normal strains are assumed to vanish.) However, rather than wait until we have introduced constitutive relations, we shall, equivalently, henceforth assume that $\delta\gamma = \mathbf{0}$. It now follows from (6b) and (22) that

$$\delta\omega_{,s} = \bar{\mathbf{e}}_r \psi' \delta\bar{\alpha} - \bar{\mathbf{e}}_\theta \delta\bar{\alpha}' + \mathbf{e}_z \delta\psi', \quad \delta\omega_{,\theta} = \bar{\mathbf{e}}_r \delta\bar{\alpha} \tag{23}$$

so that (20) reduces to

$$IVW = 2\pi \int_{S_1}^{S_2} r(N_s \delta\Lambda_s + S\delta\Gamma + N_\theta \delta\Lambda_\theta + M\delta\kappa + W\delta\tau + M_\theta \delta\kappa_\theta + D_s \delta L_s + D_\theta \delta L_\theta)\, ds, \tag{24}$$

where

$$N_s \equiv \mathbf{N}_s \cdot \hat{\mathbf{t}}, \quad S \equiv \mathbf{N}_s \cdot \bar{\mathbf{e}}_\theta, \quad N_\theta \equiv \mathbf{N}_\theta \cdot \bar{\mathbf{e}}_\theta \tag{25}$$

are stress resultants,

$$M \equiv -\mathbf{M}_s \cdot \bar{\mathbf{e}}_\theta, \quad W \equiv \mathbf{M}_s \cdot \hat{\mathbf{t}}, \quad M_\theta \equiv \mathbf{M}_\theta \cdot \hat{\mathbf{t}}, \qquad (26)\dagger$$

are stress couples,

$$D_s \equiv \mathbf{M}_s \cdot \bar{\mathbf{n}}, \quad D_\theta \equiv \mathbf{M}_\theta \cdot \bar{\mathbf{n}} \tag{27}$$

are *drilling* couples,

$$\kappa \equiv (\bar{\alpha} - \alpha)', \quad \tau \equiv \psi' \sin\bar{\alpha}, \quad \kappa_\theta \equiv r^{-1}(\sin\bar{\alpha} - \sin\alpha), \qquad (28a\text{–}c)$$

are bending strains and

$$L_s \equiv \psi' \cos\bar{\alpha}, \quad L_\theta \equiv r^{-1}(\cos\bar{\alpha} - \cos\alpha) \tag{29}$$

are *drilling* strains. The torsional bending strain, τ, was introduced by Taber (1988), but in a different way from here; κ and κ_θ are the same as in Reissner's (1950) theory of axishells.

By means of (15), (28), and (29), we have expressed all strains in terms of the four kinematic unknowns, Λ_s, Λ_θ, $\bar{\alpha}$, and ψ'. A complete set of field equations emerges when we adjoin stress–strain relations. However, before doing so, we reduce and simplify the equilibrium equations, (16) and (17), and the external virtual work, (19).

† W is a mnemonic for "wrenching" moment.

5. FIRST INTEGRALS AND OTHER SIMPLIFICATIONS OF THE EQUILIBRIUM EQUATIONS

Noting (25)–(27), we now introduce the following component representations—and alternatives—for the various vector stress resultants and couples:

$$\begin{aligned}\mathbf{N}_s &= N_s(s)\hat{\mathbf{t}}(s,\theta)+S(s)\bar{\mathbf{e}}_\theta(s,\theta)+Q(s)\bar{\mathbf{n}}(s,\theta)\\ &= H(s)\bar{\mathbf{e}}_r(s,\theta)+S(s)\bar{\mathbf{e}}_\theta(s,\theta)+V(s)\mathbf{e}_z\end{aligned}\tag{30}$$

$$\begin{aligned}\mathbf{N}_\theta &= N_{\theta s}(s)\hat{\mathbf{t}}(s,\theta)+N_\theta(s)\bar{\mathbf{e}}_\theta(s,\theta)+Q_\theta(s)\bar{\mathbf{n}}(s,\theta)\\ &= H_\theta(s)\bar{\mathbf{e}}_r(s,\theta)+N_\theta(s)\bar{\mathbf{e}}_\theta(s,\theta)+V_\theta(s)\mathbf{e}_z\end{aligned}\tag{31}$$

$$\mathbf{M} = W(s)\hat{\mathbf{t}}(s,\theta)+D_s(s)\bar{\mathbf{n}}(s,\theta)-M(s)\bar{\mathbf{e}}_\theta(s,\theta)\tag{32}$$

$$\mathbf{M}_\theta = M_\theta(s)\hat{\mathbf{t}}(s,\theta)+D_\theta(s)\bar{\mathbf{n}}(s,\theta)-M_{\theta s}(s)\bar{\mathbf{e}}_\theta(s,\theta).\tag{33}$$

Inserting the second lines of (30) and (31) into (16) and noting (6), we obtain three scalar equations which may be satisfied identically by setting

$$N_\theta = F'-G\psi',\quad H_\theta = -(G'+F\psi'),\quad rV = P,\quad \text{a constant,}\tag{34a–c}$$

where $2\pi P$ is the net (vertical) force in the z-direction acting on any section s = constant of the shell and

$$F \equiv rH,\quad G \equiv rS.\tag{35a, b}$$

Turning to moment equilibrium, we insert the second lines of (30) and (31) along with (32) and (33) into (17). Again noting (6) and using (34c) and (35), we obtain the following scalar equations in the directions of $\bar{\mathbf{e}}_r$, $\bar{\mathbf{e}}_\theta$, and $\mathbf{e}_z$, respectively:

$$r\Lambda_\theta V_\theta = \Lambda_s G\sin\bar{\alpha}-\Gamma P+[r(D_s\sin\bar{\alpha}-W\cos\bar{\alpha})]'-rM\psi'-M_{\theta s}\tag{36}$$

$$(rM)'-(M_\theta+rW\psi')\cos\bar{\alpha}+(D_\theta+rD_s\psi')\sin\bar{\alpha}+\Lambda_s(P\cos\bar{\alpha}-F\sin\bar{\alpha}) = 0\tag{37}$$

$$[r(W\sin\bar{\alpha}+D_s\cos\bar{\alpha})]'+(r\Lambda_\theta\psi'-\Gamma)F+r\Lambda_\theta G'+\Lambda_s G\cos\bar{\alpha} = 0.\tag{38}$$

We satisfy (36) identically by using it to compute V_θ. To simplify (38), we use the compatibility conditions, (14) and (15), whereby the F-term disappears while the G-terms combine into a total derivative, yielding the first integral

$$r(W\sin\bar{\alpha}+D_s\cos\bar{\alpha}+\Lambda_\theta G) = T,\quad \text{a constant,}\tag{39}$$

where $2\pi T$ is equal to the net torque about the z-axis, over any section s = constant.

By (5), (6), (12), (23), (30), (32), (34c), and (39), the external virtual work, (19), reduces to

$$EVW = 2\pi[r(F\delta\Lambda_\theta+M\delta\bar{\alpha})+P\delta z+T\delta\psi]_{S_1}^{S_2}.\tag{40}$$

6. STRESS-STRAIN RELATIONS

We now assume that the shell is elastic and, as is customary, that there are no drilling moments. That is, we assume that there exists a strain-energy per unit area of $\mathscr{S}$ of the form

$$V = V(\Lambda_s, \Lambda_\theta, \Gamma, \kappa, \kappa_\theta, \tau) \tag{41}$$

such that the variation rV equals the integrand on the right side of (24). By (24) and (40), the Principle of Virtual Work, (18), now takes the more explicit form

$$[r(F\delta\Lambda_\theta + M\delta\bar{\alpha}) + P\delta z + T\delta\psi)]_{s_1}^{s_2}$$
$$= \int_{s_1}^{s_2} (V_{,\Lambda_s}\delta\Lambda_s + V_{,\Lambda_\theta}\delta\Lambda_\theta + V_{,\Gamma}\delta\Gamma + V_{,\kappa}\delta\kappa + V_{,\kappa_\theta}\delta\kappa_\theta + V_{,\tau}\delta\tau)r\,\mathrm{d}s. \tag{42}$$

In obtaining the Euler equations implied by (42), we must recognize that $\delta\Lambda_s$, $\delta\Lambda_\theta$, and $\delta\Gamma$ are not independent, but must satisfy constraints implied by the compatibility conditions, (14) and (15). We satisfy (15) by simply replacing Γ by $r\Lambda_\theta\psi'$ wherever it appears; we satisfy (14) by means of a Lagrange multiplier which turns out to be F. That is, we add to the right side of (42) the term

$$\delta\int_{s_1}^{s_2} F[(r\Lambda_\theta)' - \Lambda_s\cos\bar{\alpha}]\,\mathrm{d}s. \tag{43}$$

Finally, noting from (10) and (11) that $z' = \Lambda_s\sin\bar{\alpha}$, we rewrite one term in the external virtual work as follows:

$$[P\delta z]_{s_1}^{s_2} = \int_{s_1}^{s_2} P(\sin\bar{\alpha}\delta\Lambda_s + \Lambda_s\cos\bar{\alpha}\delta\bar{\alpha})\,\mathrm{d}s. \tag{44}$$

We now substitute the right sides of (28) into the above-modified form of (42) and integrate by parts to remove derivatives of variations. First setting the coefficient of $\delta\bar{\alpha}$ to zero in the interior and on the boundary of the shell, we obtain

$$(rV_{,\kappa})' = (V_{,\kappa_\theta} + r\chi V_{,\tau})\cos\bar{\alpha} + \Lambda_s(F\sin\bar{\alpha} - P\cos\bar{\alpha}), \quad s_1 < s < s_2 \tag{45*}$$

and

$$M = V_{,\kappa}, \quad s = s_1, s_2, \tag{46}$$

where

$$\chi \equiv \psi'. \tag{47}$$

If we rewrite (28a) in the form of a first-order differential equation,

$$\bar{\alpha}' = \alpha' + \kappa, \quad s_1 < s < s_2, \tag{48*}$$

then, by (15), (28b,c), and (47), the strain-energy density, V, given by (41), takes the functional form

$$V = \tilde{V}(\Lambda_s, \Lambda_\theta, \bar{\alpha}, \kappa, \chi) \tag{49}$$

and (45) becomes a first-order differential equation (involving, in general, derivatives of all the unknowns except F).

Next, we consider the coefficient of $\delta\Lambda_\theta$. Assuming for simplicity that either H (and hence F) or $\bar{r}$ (and hence Λ_θ) is prescribed on the boundary of the shell, we obtain the differential equation

$$F' = V_{,\Lambda_\theta} + r\chi V_{,\Gamma}, \quad s_1 < s < s_2 \tag{50*}$$

and boundary conditions

$$F = \hat{F} \quad \text{or} \quad \Lambda_\theta = \hat{\Lambda}_\theta, \quad s = s_1, s_2, \tag{51}$$

where a hat (ˆ) denotes a prescribed quantity.

Setting the coefficient of δF to zero yields, by (43), the compatibility condition, (14).

Setting the coefficient of $\delta\psi$ to zero in the interior yields an equation of the form $(\ldots)' = 0$ which, if we integrate and note the term $[T\delta\psi]_{s_1}^{s_2}$ on the left side of (42), implies the *algebraic* relation

$$r(V_{,\tau}\sin\bar{\alpha} + r\Lambda_\theta V_{,\Gamma}) = T. \tag{52)*$$

Our final Euler relation,

$$F\cos\bar{\alpha} + P\sin\bar{\alpha} = rV_{,\Lambda_s}, \tag{53)*$$

is also an algebraic relation and follows from equating to zero the coefficient of $\delta\Lambda_s$.

Comparing the equations of this section with those of Section 5, we have the stress–strain relations

$$N_s = V_{,\Lambda_s}, \quad N_\theta = V_{,\Lambda_\theta}, \quad S = V_{,\Gamma}, \quad M = V_{,\kappa}, \quad M_\theta = V_{,\kappa_\theta}, \quad W = V_{,\tau}. \tag{54}$$

In summary, the final form of our field equations are the four first-order differential equations, (14), (45), (48), and (50), plus the two algebraic relations (52) and (53). To solve them, the strain-energy density, V, must be specified. For a neo-Hookean shell, we may use the expression derived by Taber (1988, eqn 55) which, under the Kirchhoff hypothesis and in our notation, takes the form

$$V = \tfrac{1}{2}\mu h\left\{\Lambda_s^2 + \Lambda_\theta^2 + \frac{1}{\Lambda_s^2\Lambda_\theta^2} + \Gamma^2 - 3 + \frac{h^2}{12\Lambda_s^2\Lambda_\theta^2}\left[\left(1 + \frac{3}{\Lambda_s^4\Lambda_\theta^2}\right)\kappa^2 + \frac{4\kappa\kappa_\theta}{\Lambda_s^3\Lambda_\theta^3} + \left(1 + \frac{3}{\Lambda_s^2\Lambda_\theta^4}\right)\kappa_\theta^2 + \tau^2\right]\right\}, \tag{55}$$

where μ is a shear modulus.

7. SIMPLIFICATIONS IMPLIED BY A QUADRATIC STRAIN-ENERGY DENSITY

If Γ and the "engineering" meridional and hoop strains

$$\Gamma_s \equiv \Lambda_s - 1 \quad \text{and} \quad \Gamma_\theta \equiv \Lambda_\theta - 1 \tag{56}$$

are sufficiently small, then the strain-energy density of an elastically isotropic shell, free of initial stress, may be represented in the form (Koiter, 1960)

$$V = \tfrac{1}{2}C[\Gamma_s^2 + 2\nu_s\Gamma_s\Gamma_\theta + \Gamma_\theta^2 + \underline{\tfrac{1}{2}(1-\nu_s)\Gamma^2}] + \tfrac{1}{2}D[\kappa^2 + 2\nu_b\kappa\kappa_\theta + \kappa_\theta^2 + \underline{\tfrac{1}{2}(1-\nu_b)\tau^2}] + R. \tag{57}$$

Here, C and D are stretching and bending stiffnesses, ν_s and ν_b are Poisson ratios of stretching and bending, R is a certain remainder (or error) term, and the underlines indicate terms absent in shells of revolution undergoing torsionless, axisymmetric deformation (axishells). It is conventional (but not necessary) to take

$$\nu_s = \nu_b = \nu, \quad C = \frac{Eh}{1-\nu^2}, \quad D = \frac{Eh^3}{12(1-\nu^2)}, \tag{58}$$

where E is Young's modulus and h is the undeformed thickness of the shell, here assumed constant. The associated stress–strain relations are, from (54) and (56),

$$N_s = C(\Gamma_s + \nu_s \Gamma_\theta), \quad N_\theta = C(\Gamma_\theta + \nu_s \Gamma_s), \quad \underline{S = \tfrac{1}{2}(1-\nu_s)C\Gamma} \tag{59}$$

and

$$M = D(\kappa + \nu_b \kappa_\theta), \quad M_\theta = D(\kappa_\theta + \nu_b \kappa), \quad \underline{W = \tfrac{1}{2}(1-\nu_b)D\tau}. \tag{60}$$

Solving (59) for strains in terms of stresses and introducing (34a), (35), and the representation

$$rN_s = F\cos\bar{\alpha} + P\sin\bar{\alpha} \tag{61}$$

which follows from (30), (34c), and (35a), we have

$$r\Gamma_s = A[F\cos\bar{\alpha} + P\sin\bar{\alpha} - \nu_s r(F' - \underline{G\chi})]$$

$$r\Gamma_\theta = A[r(F' - \underline{G\chi}) - \nu_s(F\cos\bar{\alpha} + P\sin\bar{\alpha})], \quad \underline{r\Gamma = 2(1+\nu_s)AG}, \tag{62a-c}$$

where A is a stretching compliance, conventionally taken as

$$A = \frac{1}{Eh}. \tag{63}$$

Simmonds and Libai (1987) (see also Libai and Simmonds, 1988, Section V.R) have simplified Reissner's (1972) equations for axishells by excluding terms that are of the same order of magnitude as those that would have appeared had certain of the R-terms in (57) been kept, such as those representing transverse shearing strains. Adding to eqns (40) and (41) of Simmonds and Libai (1987)—which correspond to (37) or (45) and (14), respectively—those terms coming from the underlined terms in (60) and (62), we obtain

$$D[(r\beta')' - r^{-1}\sin\beta - \underline{(1/4)(1-\nu_b)r\chi^2\sin 2(\alpha+\beta)}] + P\cos(\alpha+\beta) - F\sin(\alpha+\beta) = 0 \tag{64}$$

and

$$A[(rF')' - r^{-1}F - \underline{\nu_s G\chi}] + \cos\alpha - \cos(\alpha+\beta) = 0, \tag{65}$$

where $\beta = \bar{\alpha} - \alpha$.

Two addition equations relating χ and G follow from (15), (28b), (35b), (52), (57), and (62c) as

$$2(1+\nu_s)AG = r^2\chi \tag{66}$$

and

$$r[\tfrac{1}{2}(1-\nu_b)D\chi\sin^2(\alpha+\beta) + G] = T. \tag{67}$$

Here, we have neglected strains compared to unity, consistent with ignoring the R-terms in the expression for the strain-energy density, (57). Solving these two equations for G, we have

$$G = \frac{T}{r}\left[1 + O\left(\frac{h^2\sin^2\bar{\alpha}}{r^2}\right)\right]. \tag{68}$$

We note that the second term in the brackets is negligible, except possibly near the axis of revolution, so we shall neglect it. Inserting (66) and (68) into (64) and (65), we obtain the final form of our simplified equations:

$$D[(r\beta')' - r^{-1}\sin\beta] + P\cos(\alpha+\beta) - F\sin(\alpha+\beta) - \underline{\tfrac{1}{2}(1-\nu_b)D[(1+\nu_s)AT/r^2]^2 r^{-1}\sin 2(\alpha+\beta)} = 0 \quad (69)$$

$$A[(rF')' - r^{-1}F] + \cos\alpha - \cos(\alpha+\beta) = \underline{2\nu_s(1+\nu_s)(AT/r^2)^2}. \quad (70)$$

8. CONCLUSIONS

Our equations can be extended in several obvious ways. First, surface loads can be incorporated easily. Second, we can consider a polar orthotropic material in which the thickness varies with undeformed meridional distance, s. And third, we can follow Reissner (1968) and generalize the parametric equation for the deformed reference surface, (5), to encompass helical deformations of incomplete shells of revolution. Such extensions would complicate but not change the basic structure of our simplified equations of Section 6 for large strains or Section 7 for small strains.

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